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Oscillation Spaces

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Dual space characterizations are given for weak Haar, Haar, and oscillating subspaces of real valued (not necessarily continuous) functions on the line. The characterizations improve known results, and provide a unified simpler approach to oscillation spaces. © 1992 Academic Press, Inc.

INTRODUCTION

If F is an n dimensional Haar space, then by definition no function in F can have n sign changes. However, it may be possible for the "direction" of a function f to have n sign changes. That is, there may be points $x_1 < \cdots < x_{n+1}$ such that $(-1)^i [f(x_{i+1}) - f(x_i)] > 0$ for $i = 1, \ldots, n$. There are some beautiful theorems about this phenomenon. For example, if F is a normalized Markov space (i.e., there exist Haar spaces F_i of dimension i such that F_1 is the constant, and $F_1 \subset \cdots \subset F_{n-1} \subset F$) then no function in F can have such an oscillation. The circle of ideas involves four kinds of spaces: Haar, weak Haar, strong oscillation, and oscillation spaces.

This paper has two purposes. The first is to present new results about oscillation spaces. These include characterizations of these spaces in terms of the functionals that annihilate them, and an annihilator lifting theorem. It also includes improving known results. For example, the common hypothesis that an oscillation space contains the constants is shown to be superfluous, and the "betweeness" property (B) for the domain **X** of functions in the Markov Characterization of strong Haar spaces is reduced to the much weaker assumption that **X** has no two point end sets.

The second purpose is to show that the annihilator characterizations provide a unified approach to the main results (such as the determinant and Markov characterizations) of oscillations. The characterizations replace ad hoc computational case tracking proofs with strikingly economical "soft analysis" arguments. A final "Notes" section contains comparisons to the literature.

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Notation

Let $\mathbf{X}$ be a subset of the real line. Let Δ_k be the collection of all subsets of $\mathbf{X}$ consisting of k distinct points listed in their natural order.

Measures. For $x \in \mathbf{X}$, let $\hat{x}$ be the point evaluation functional at x. That is, for every real function f on $\mathbf{X}$, $\hat{x}(f) = f(x)$. For x and $y \in \mathbf{X}$ put $\alpha(x, y) = \hat{x} - \hat{y}$. For $Y = \{x_1, ..., x_k\} \in \Delta_k$, define $\alpha_i = \alpha(x_i, x_{i-1})$, $i = 2, ..., k$, and put

$$S(Y) = \text{convex hull}\{\{(-1)^i \hat{x}_i\}_{i=1}^k\},$$

$$P(Y) = \left\{\sum_{i=1}^k c_i \hat{x}_i : c_i \neq 0,\ i = 1, 2, ..., k\right\},$$

$$H(Y) = \text{convex hull}\{\{(-1)^i \alpha_i\}_{i=2}^k\},$$

and

$$Q(Y) = \left\{\sum_{i=2}^k (-1)^i c_i \alpha_i : c_i > 0,\ i = 2, ..., k\right\} \cap H(Y).$$

We note that each member of $H(x_1, ..., x_k)$ is a positive multiple of a member of $S(x_1, ..., x_k)$. If F is a linear space F^* is the space of linear functionals on F. For $E \subset F$, $E^\perp = \{L \in F^* : L(g) = 0 \text{ for all } g \in E\}$. Such functionals are called *annihilators* of E. The dimension of a linear space E is written dim E.

Functions. All functions will be real valued functions on $\mathbf{X}$. We do not assume that the functions are continuous. A function f is said to have an *alternation of length* k if there is a $\{x_1, ..., x_k\} \in \Delta_k$ such that f (or possibly $-f$) satisfies $(-1)^{i-1} f(x_i) > 0$. It is said to have an *oscillation* (*weak oscillation*, resp.) *of length* k if f (or $-f$) satisfies $(-1)^i [f(x_{i+1}) - f(x_i)] > 0$ ($\geqslant 0$, resp.).

The Dirac delta function is the function $\delta_{i,j}$ which is 1 if $i = j$, and 0 if $i \neq j$. The restriction of a family of functions F to a set Y is written $F|_Y$. The sign of a number c is -1 if $c < 0$, it is 0 if $c = 0$, and 1 if $c > 0$. We say that numbers a and b have *weakly equal sign* and write sign $a \cong$ sign b if a and b are both nonnegative or both nonpositive.

For $f_1, f_2, ..., f_n$ a basis for F, $(x_1, ..., x_n) \in \Delta_n$, we put

$$\det\{f_i(x_j)\} = \begin{vmatrix} f_1(x_1) & f_1(x_2) & \cdots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(x_1) & f_n(x_2) & \cdots & f_n(x_n) \end{vmatrix}.$$

Spaces. Let F be an n-dimensional space of real valued functions on $\mathbf{X}$. F is called *Chebyshev* if 0 is the only member of F with n zeros. It is termed a *weak Haar space* if none of its members has an alternation of length $n+1$. If F is both Chebyshev and weak Haar it is a *Haar space.* F is called an *oscillating* (*strong oscillation*, resp.) *space* if no $f \in F$ (nonconstant $f \in F$ resp.) has an oscillation (weak oscillation, resp.) of length $n+1$. It is clear that a strong oscillating space is a Haar space. We afix the title *Markov* to a space if it contains a family of nested subspaces—one for each dimension—which share the same property. For example, F is Markov Haar space if there are subspaces $F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F$ such that each F_i is a Haar space of dimension i. If, in addition, the one dimensional subspace, of a Markov space F, is the space of constant functions we call F a *normalized* Markov space.

Conventions. Throughout this paper we will let F represent an n-dimensional space of real valued functions defined on $\mathbf{X}$. We will often use the same symbol for a measure and for the linear functional associated with it.

1. Preliminary Lemmas

We list, in this section, some elementary results, mainly from linear algebra, that we reference later.

Definition. For Y in the domain of F, and $p \in Y$ we say that p is *independent* (of Y, and with respect to F) if $\dim F|_Y > \dim F|_{Y-\{p\}}$. We call a set $Y \subseteq \mathbf{X}$ *basic* (for F) if $\dim F|_Y = n$. We put

$$(\mathbf{B}\Delta)_k = \{Y \in \Delta_k \colon Y \text{ is basic}\}.$$

Lemma 1.1. *p is independent of a set Y with respect to F if and only if there is an $f \in F$ that vanishes on $Y - \{p\}$ and such that $f(p) = 1$.*

Proof. $\dim F$ decreases on $Y - \{p\}$, if and only if there is a nontrivial linear combination of members of F that is zero on $Y - \{p\}$. ∎

Lemma 1.2. *Let $0 \neq v = \sum_{i=1}^{n+1} v_i \hat{x}_i \in F^{\perp}$. x_j is independent of $\{x_i\}_{i=1}^{n+1}$ with respect to F, implies $v_j = 0$. Furthermore if $\{x_i\}_{i=1}^{n+1} \in (\mathbf{B}\Delta)_{n+1}$, then $v_j = 0$ implies that x_j is independent.*

Proof. If x_j is independent there is a $g \in F$ that is nonzero only at x_j. So $v(g) = 0$ implies that $v_j = 0$. Conversely suppose that $v_j = 0$, we show that g is in F. Since F has codimension 1 in the set of all functions on $\{x_i\}_{i=1}^{n+1}$, we have that $F = v^{-1}(0)$. Since $g \in v^{-1}(0)$, g is in F.

LEMMA 1.3. *F is weak Haar (oscillating, resp.) if and only if $F|_Y$ is weak Haar (oscillating, resp.) for each $Y \in (\mathbf{B}\Delta)_{n+1}$.*

Proof. The proof for oscillating spaces is precisely the same as that for the weak Haar with the word "alternation" (which appears three times below) replaced by "oscillation." Suppose that F is not weak Haar. There is an $f \in F$ which has an alternation of length $n+1$ on some set $\{x_i\}_{i=1}^{n+1}$. If this set is not basic, we will substitute a new point for one of its members in such a way that the dimension of F on the new set exceeds that on the old, and such that F still has a function with an $n+1$ alternation on the new set. So now assuming that $\{x_i\}_{i=1}^{n+1}$ is not basic, from Lemma 1.1 there is a $t \in \mathbf{X}$ and $g \in F$ such that $g(x_i)=0$, $i=1, ..., n+1$, and $g(t)=1$. We assume that $x_k < t < x_{k+1}$ where $x_0 = -\infty$, and $x_{n+2} = \infty$. We will assume that $t < x_{n+1}$ and substitute t for a member to its right. (Otherwise we use the same procedure and replace a member of its left.) Let x_w be the first member to the right of t that is not independent (of $\{x_i\}_{i=1}^{n+1}$ with respect to $F|_{\{x_i\}_{i=1}^{n+1}}$). If no such point exists then let $w=n+1$. By Lemma 1.1, choose for each $j=k+1, ..., w-1$ and f_j that vanishes on $\{x_i\}_{i\neq j}$ and is 1 at x_j. Then

$$f+[f(x_{k+1})-f(t)]\, g+\sum_{j=k+2}^{w}[f(x_j)-f(x_{j-1})]\, f_{j-1}$$

has the same values on $x_1, ..., t, x_{k+1}, ..., x_{w-1}, x_{w+1}, ..., x_{n+1}$ as f on $x_1, ..., x_{n+1}$. Consequently both have alternations of length $n+1$. ∎

LEMMA 1.4. *Suppose that f has an (weak, resp.) oscillation of length $n+1$ on $\{x_i\}_{i=1}^{n+1} \in \Delta_{n+1}$. If $x_0 < x_1$ then there are $\{y_i\}_{i=1}^{n} \in \Delta_n$ such that $x_0 < y_1$ and f has an (weak, resp.) oscillation of length $n+1$ on $\{x_0, y_1, y_2, ..., y_n\}$.*

Proof. Suppose $f(x_1) > f(x_2)$ (or $\geqslant$ if f is weak oscillating). Then if $f(x_0) > f(x_1)$, put $y_i = x_{i+1}$. If $f(x_0) \leqslant f(x_1)$, put $y_i = x_i$. ∎

2. CHARACTERIZATIONS

THEOREM 2.1 (Annihilator Characterization). (a) *F is oscillating if and only if for every $\mathbf{x} \in \Delta_{n+1}$, $H(\mathbf{x}) \cap F^1 \neq \varnothing$.*

(b) *F is a weak Haar space if and only if for every $\mathbf{x} \in \Delta_{n+1}$, $S(\mathbf{x}) \cap F^{\perp} \neq \varnothing$.*

(c) *F is a strong oscillation space if and only if for every $\mathbf{x} \in \Delta_{n+1}$, $Q(\mathbf{x}) \cap F^{\perp} \neq \varnothing$.*

(d) *F is Chebyshev if and only if every annihilator of F that is supported on $\{x_1, ..., x_{n+1}\} \in \Delta_{n+1}$ is in $P(x_1, ..., x_{n+1})$.*

Proof. Suppose $f \in F$ has an oscillation of length $n+1$ on $\{x_1, ..., x_{n+1}\} \in \Delta_{n+1}$. We can assume that $f(x_1) < f(x_2)$. Let $\mu = \sum_{i=2}^{n} (-1)^i c_i \alpha_i \in H(x_1, ..., x_{n+1})$. For each i, $(-1)^i \alpha_i f > 0$. Since not each c_i is zero, $\mu f > 0$ and μ is not in $F^\perp$.

Conversely suppose there does exist $\{x_1, ..., x_{n+1}\} \in \Delta_{n+1}$ such that $H(x_1, ..., x_{n+1}) \cap F^\perp = \varnothing$. From the Hahn–Banach Theorem there is an $f \in F$ such that $\mu(f) > 0$ for all $\mu \in H(x_1, ..., x_{n+1})$. In particular $(-1)^i \alpha_i(f) > 0$ for each i. But this says that f has an oscillation of length $n+1$. This proves part (a).

The proof above for oscillation spaces adapts to the weak Haar setting by replacing the word "oscillation" with "alternation," and replacing the set $H(x_1, ..., x_{n+1})$ with $S(x_1, ..., x_{n+1})$.

Just as in the proof of (a) above $Q(\mathbf{x}) \cap F^\perp \neq \varnothing$ for all $\mathbf{x} \in \Delta_{n+1}$ implies that F is a strong oscillation space. So now suppose that F is a strong oscillation space. For $\{x_i\}_{i=1}^{n+1} \in \Delta_{n+1}$ there are nonnegative numbers $\{c_i\}_{i=2}^{n+1}$ such that $\mu = \sum_{i=2}^{n+1} (-1)^i c_i \alpha_i \in F^\perp$. We need to show that each $c_i > 0$. Since F is a Haar space, its restriction to $\{x_i\}_{i=1}^{n+1}$ is still n-dimensional. Hence it is all of the null space of μ. Suppose that $c_k = 0$. Let $g(x_i)$ be 0 for $i < k$ and 1 if $k \leqslant i$. Then $\mu(g) = 0$, and so it is the restriction to $\{x_i\}_{i=1}^{n+1}$ of some member of F. However, g has a weak oscillation of length $n+1$. This proves (c).

Part (d) follows from the fact that F is Chebyshev if and only if the functionals associated with any set of n distinct points of $\mathbf{X}$ are linearly independent. ▮

COROLLARY 2.2. *An oscillating space F contains the constants.*

Proof. If $F \cup \{1\}$ is $n+1$ dimensional, there are points $\{x_i\}_{i=1}^{n+1}$ so that the restriction of $F \cup \{1\}$ to $\{x_i\}_{i=1}^{n+1}$ is $n+1$ dimensional. By the characterization Theorem 2.1 there is a $\mu \in H(x_1, ..., x_{n+1}) \cap F^\perp$. But then the null space of μ is an n dimensional subspace (of the space of all functions on $\{x_i\}_{i=1}^{n+1}$) which contains both F and the constants. ▮

COROLLARY 2.3. *Let $x_1 \in \mathbf{X}$. No member of F admits a (weak, resp.) oscillation of length $n+1$ on $[x_1, \infty) \cap \mathbf{X}$ if and only if for sets of n points $\{x_i\}_{i=2}^{n+1}$ such that $x_1 < x_2 < \cdots < x_{n+1}$, we have $H(x_1, ..., x_{n+1}) \cap F^1 \neq \varnothing$.*

Proof. Suppose that $Y = \{\{y_i\}_{i=1}^{n+1} \subset [x_1, \infty) \cap \mathbf{X}$ and $\{y_i\}_{i=1}^{n+1} \in \Delta_{n+1}\}$. By Lemma 1.4 if F restricted to each $\{y_i\}_{i=1}^{n+1} \in Y$ with $y_1 = x_1$ is an oscillation space, then F restricted to each $y \in Y$ is an (weak, resp.) oscillation space. ▮

COROLLARY 2.4 (Generic Subspaces Lemma). *Suppose that* $u<v$, *and that both are in* $\mathbf{X}$. *Suppose, also, there are g and* $h\in F$ *such that* $g(u)\neq 0$ *and* $h(u)\neq h(v)$. *Put* $E=\{f\in F: f(u)=f(v)\}$ *and* $G=\{f\in F: f(u)=0\}$.

(a) *If F is an oscillating space, then E is an* $n-1$ *dimensional oscillating space on the domain* $[v,\infty)\cap\mathbf{X}$.

(b) *If F is a strongly oscillating space, then E is an* $n-1$ *dimensional strongly oscillating space on the domain* $[v,\infty)\cap\mathbf{X}$.

(c) *If F is a weak Haar space, then G is an* $n-1$ *dimensional weak Haar space on* $[u,\infty)\cap\mathbf{X}$.

Proof. Suppose that F is oscillating. Let $\{x_3<x_4<\cdots<x_{n+1}\}\subset(v,\infty)\cap\mathbf{X}$. Put $x_1=u$ and $x_2=v$. From the oscillation space characterization Theorem 2.1, there are nonnegative constants c_i such that $\sum_{i=2}^{n+1}(-1)^i c_i\alpha_i\in F^\perp$. Since $\alpha_2\in E^\perp$, we have that $\sum_{i=3}^{n+1}(-1)^i c_i\alpha_i\in E^\perp$. By Corollary 2.3 this implies that E has no n oscillating functions on $[v,\infty)\cap\mathbf{X}$. This proves part (a).

If F is strongly oscillating all the c_i obtained in the last paragraph are positive. So E is strongly oscillating.

Assume now that F is a weak Haar space. Let $\{x_2<x_3<\cdots<x_{n+1}\}\subset(u,\infty)\cap\mathbf{X}$. Put $x_1=u$. There are nonnegative constants c_i such that $\sum_{i=1}^{n+1}(-1)^i c_i\hat{x}_i\in F^\perp$. Since $\hat{x}_1\in G^\perp$, we have that $\sum_{i=2}^{n+1}(-1)^i c_i\alpha_i\in G^\perp$. Therefore G is a weak Haar space on $(u,\infty)\cap\mathbf{X}$ and therefore on $[u,\infty)\cap\mathbf{X}$. ∎

It follows from the independence of the functionals associated with distinct points that if F is Chebyshev then $\det\{f_i(x_j)\}\neq 0$. The following similarly characterizes weak Haar spaces.

THEOREM 2.5 (Determinant Characterization). *Let* $f_1,\dots,f_n$ *be a fixed basis for F. Then F is a weak Haar space if and only if* $\det\{f_i(x_j)\}$ *weakly has the same sign (i.e., is always nonpositive or is always nonnegative) for each* $(x_1,\dots,x_n)\in\Delta_n$.

Proof. Suppose first that F has the weak constant sign property for the determinants. For $\{x_i\}_{i=1}^{n+1}\in(\mathbf{B}\Delta)_{n+1}$, we wish to show that $S(x_1,\dots,x_{n+1})\cap F^\perp\neq\varnothing$. It would then follow from the Annihilator Characterization Theorem 2.1(b) and Lemma 1.3 that F is a weak Haar space.

Since $\{x_i\}_{i=1}^{n+1}$ is basic with respect to F, some set of n of the associated point evaluation functionals $\hat{x}_i$ are linearly independent on F. For specificity we will assume that $\{\hat{x}_i\}_{i=1}^{n}$ are linearly independent (keeping track of the indicies in the general argument obscures the simple idea of the proof).

We choose a basis $\{f_i\}_{i=1}^n$ of F with the property that $f_i(x_j)=\delta_{i,j}$ for $i, j=1, ..., n$. Then by the sign property for the determinants associated with $\{x_1, ..., x_n\}$ and $\{x_1, ..., x_{n-1}, x_{n+1}\}$

$$\operatorname{sign}\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix} \cong \operatorname{sign}\begin{vmatrix} 1 & 0 & \cdots & 0 & f_1(x_{n+1}) \\ 0 & 1 & \cdots & 0 & f_2(x_{n+1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & f_{n-1}(x_{n+1}) \\ 0 & 0 & \cdots & 0 & f_n(x_{n+1}) \end{vmatrix}.$$

Computing the second determinant by expanding in cofactors along the bottom row gives $f_n(x_{n+1})\geqslant 0$. From the definition of f_n, $[f_n(x_{n+1})\,\hat{x}_n-\hat{x}_{n+1}](f_n)=0$.

Similarly for the determinant associated with $x_1, ..., x_{n-2}, x_n, x_{n+1}$ we have

$$\operatorname{sign}\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix} \cong \operatorname{sign}\begin{vmatrix} 1 & \cdots & 0 & 0 & f_1(x_{n+1}) \\ \vdots & \ddots & & & \vdots \\ 0 & \cdots & 1 & 0 & f_{n-2}(x_{n+1}) \\ 0 & \cdots & 0 & 0 & f_{n-1}(x_{n+1}) \\ 0 & \cdots & 0 & 1 & f_n(x_{n+1}) \end{vmatrix}$$

$$= -\operatorname{sign}\begin{vmatrix} 1 & \cdots & 0 & f_1(x_{n+1}) & 0 \\ \vdots & \ddots & & \vdots & \\ 0 & \cdots & 1 & f_{n-2}(x_{n+1}) & 0 \\ 0 & \cdots & 0 & f_{n-1}(x_{n+1}) & 0 \\ 0 & \cdots & 0 & f_n(x_{n+1}) & 1 \end{vmatrix}.$$

Hence $f_{n-1}(x_{n+1})\leqslant 0$ and $[f_{n-1}(x_{n+1})\,\hat{x}_{n-1}-\hat{x}_{n+1}](f_{n-1})=0$.

Continuing this process we have that $(-1)^{n-j}f_j(x_{n+1})\leqslant 0$, and $\sum_{i=1}^n f_i(x_{n+1})\,\hat{x}_i-\hat{x}_{n+1}$ annihilates each f_i and so a positive multiple of it belongs to $S(x_1, ..., x_n)\cap F^{\perp}$, and F is a weak Haar space.

Now suppose that F is a weak Haar space, and has a basis $f_1, ..., f_n$. Let $\{x_i\}_{i=1}^{n+1}\in\Delta_{n+1}$. For each $1\leqslant i\leqslant n+1$, let $i_k=k$ for $k<i$, and $=k+1$ for $k\geqslant i$. We first show that the determinants $D(i)=\det\{f_j(x_{i_k})\}$ weakly have the same sign. If all these determinants are zero then the statement is true. Again to avoid the obfuscation of index tracking in the general case we will assume that $D(n+1)\neq 0$. Since F is assumed to be a weak Haar space there are $c_i\geqslant 0$ such that $\sum_{i=1}^{n+1}(-1)^i c_i\hat{x}_i\in F^{\perp}$. $D(n+1)\neq 0$ implies that $c_{n+1}\neq 0$. So on F, $\sum_{j=1}^n((-1)^j c_j/(-1)^n c_{n+1})\,\hat{x}_j=\hat{x}_{n+1}$. Now for a fixed i $(1\leqslant i\leqslant n)$, using elementary column operations to replace the ith column, we have

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & \cdots & \sum_{j=1}^{n} \frac{(-1)^j c_j}{(-1)^n c_{n+1}} f_1(x_j) & \cdots & f_1(x_n) \\ \vdots & & & \vdots & & \vdots \\ f_n(x_1) & f_n(x_2) & \cdots & \sum_{j=1}^{n} \frac{(-1)^j c_j}{(-1)^n c_{n+1}} f_n(x_j) & \cdots & f_n(x_n) \end{vmatrix}$$

$$= \frac{(-1)^i c_i}{(-1)^n c_{n+1}} \begin{vmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{vmatrix}$$

$$= (-1)^{n-i} \frac{c_i}{c_{n+1}} D(n+1).$$

But the first determinant is equal to

$$\begin{vmatrix} f_1(x_1) & f_1(x_1) & \cdots & f_1(x_{n+1}) & \cdots & f_1(x_n) \\ \vdots & & & \vdots & & \vdots \\ f_n(x_1) & f_n(x_1) & \cdots & f_n(x_{n+1}) & \cdots & f_n(x_n) \end{vmatrix} = (-1)^{n-i} D(i).$$

We conclude that sign $D(n+1) \cong$ sign $D(i)$.

Finally suppose that $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ are both in $(\mathbf{B}\Delta)_n$ (with respect to F). We want to show that $\operatorname{sign} \det\{f_j(x_i)\}_{i,j=1}^n = \operatorname{sign} \det\{f_j(y_i)\}_{i,j=1}^n$. The above paragraph shows that if we replace the x_i's one at a time by y_i's, the resulting new determinants weakly have the same sign as its predecessor. We need to observe that this can be done with none of the determinants being equal to zero. We do this as follows. Since $\{\hat{x}_i\}_{i=1}^n$ is a basis for the dual of F, some (unique and nonzero) linear combination of them equals $\hat{y}_1$ (on F). Use y_1 to replace any one of the x_i that has a nonzero coefficient. The resulting new set in Δ_n still produces a nonzero determinant. Having replaced $k-1$ members of $\{x_i\}_{i=1}^n$ with $\{y_i\}_{i=1}^{k-1}$, to produce a new basic set B we write $\hat{y}_k$ as a linear combination of the members of B. One of the functionals with a nonzero coefficient must be from the set $\{\hat{x}_i\}_{i=1}^n$ (since $\{\hat{y}_i\}_{i=1}^n$ is linearly independent). Hence replacing that element with y_k completes the induction step. ∎

COROLLARY 2.6. *Let $f_1, ..., f_n$ be a fixed basis for F. Then F is a Haar space if and only if* $\det\{f_i(x_j)\}$ *has the same sign for each* $(x_1, ..., x_n) \in \Delta_n$.

3. LIFTING PROPERTIES

LEMMA 3.1 (Annihilator Lifting Lemma). *Let $F = \{f\} + E$ be a weak Haar space. Let both $\{x_i\}_{i=1}^n$ and $\{x_i\}_{i=2}^{n+1} \in (\mathbf{B}\Delta)_n$ with respect to F. If*

$\mu = \sum_{i=1}^{n} \mu_i \hat{x}_i \in S(x_1, ..., x_n) \cap E^{\perp}$ *and* $\nu = \sum_{i=2}^{n+1} \nu_i \hat{x}_i \in S(x_2, ..., x_{n+1}) \cap E^{\perp}$ *then* sign $\mu(f) \cong$ sign $\nu(f)$.

Proof. Let $E = \operatorname{span}\{f_1, ..., f_{n-1}\}$. There is a set of $n-1$ points in $\{x_i\}_{i=1}^{n+1}$ whose associated point evaluation functionals are linearly independent on E. Let x_s and x_t be the two points left out. We may assume that $1 \leqslant s \leqslant n$ and $2 \leqslant t \leqslant n+1$. From Lemma 1.2 we have that $\mu_s \neq 0 \neq \nu_t$, and that $\det\{f_i(x_j)\}_{i=1, j \neq s,t}^{n-1, n+1} \neq 0$. For clarity we will assume that $s = 1$ and $t = n+1$. The general case is precisely the same.

Using elementary column operations we have

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & \cdots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_n) \\ \vdots & \vdots & & \vdots \\ f_{n-1}(x_1) & f_{n-1}(x_2) & \cdots & f_{n-1}(x_n) \\ f(x_1) & f(x_2) & \cdots & f(x_n) \end{vmatrix}$$

$$= \frac{1}{\mu_1} \begin{vmatrix} \mu(f_1) & f_1(x_2) & \cdots & f_1(x_n) \\ \mu(f_2) & f_2(x_2) & \cdots & f_2(x_n) \\ \vdots & \vdots & & \vdots \\ \mu(f_{n-1}) & f_{n-1}(x_2) & \cdots & f_{n-1}(x_n) \\ \mu(f) & f(x_2) & \cdots & f(x_n) \end{vmatrix}.$$

Since $\mu \in E^{\perp}$, expanding the last derminant in cofactors along the first column gives

$$\frac{1}{\mu_1} (-1)^{n-1} \mu(f) \begin{vmatrix} f_1(x_2) & \cdots & f_1(x_n) \\ f_2(x_2) & \cdots & f_2(x_n) \\ \vdots & & \vdots \\ f_{n-1}(x_2) & \cdots & f_{n-1}(x_n) \end{vmatrix}.$$

Similarly,

$$\begin{vmatrix} f_1(x_2) & f_1(x_3) & \cdots & f_1(x_{n+1}) \\ f_2(x_2) & f_2(x_3) & \cdots & f_2(x_{n+1}) \\ \vdots & \vdots & & \vdots \\ f_{n-1}(x_2) & f_{n-1}(x_3) & \cdots & f_{n-1}(x_{n+1}) \\ f(x_2) & f(x_3) & \cdots & f(x_{n+1}) \end{vmatrix}$$

$$= \frac{1}{\nu_{n+1}} \nu(f) \begin{vmatrix} f_1(x_2) & \cdots & f_1(x_n) \\ f_2(x_2) & \cdots & f_2(x_n) \\ \vdots & & \vdots \\ f_{n-1}(x_2) & \cdots & f_{n-1}(x_n) \end{vmatrix}.$$

The left side determinants in the two equations above have weakly equal sign since F is a weak Haar space. The last determinant displayed is not zero since no function in E vanishes on $\{x_2, ..., x_n\}$. Since sign $\mu_1 = (-1)^{n-1}$ sign v_{n+1} we conclude that sign $\mu(f) = \text{sign } v(f)$. ∎

THEOREM 3.2 (Liftings). (a) *An n dimensional weak Haar space F is an oscillating space if it contains an n − 1 dimensional oscillating space E.*

(b) *An n dimensional Haar space F is a strong oscillating space if it contains an n − 1 dimensional strong oscillating space E.*

Proof. To prove (a) it suffices (from Lemma 1.3) to show that for $\{x_i\}_{i=1}^{n+1} \in (\mathbf{B}\Delta)_{n+1}$, $H(x_1, ..., x_{n+1}) \cap F^{\perp} \neq \varnothing$. Since E is an $n-1$ dimensional oscillating space on $\{x_i\}_{i=1}^{n+1}$ there are $\mu \in H(x_1, ..., x_n) \cap E^{\perp}$ and $v \in H(x_2, ..., x_{n+1}) \cap E^{\perp}$. If x_1 is independent of $\{x_i\}_{i=1}^{n+1}$ with respect to F, there exists a $g \in F$ such that $g(x_1) = 1$ and $g(x_i) = 0$ for $i = 2, ..., n+1$. Hence $v \in F^{\perp}$ and is the desired measure. Similarly if x_{n+1} is independent of $\{x_i\}_{i=1}^{n+1}$, $\mu \in F^{\perp}$ and is the desired measure. If neither x_1 nor x_{n+1} is independent of $\{x_i\}_{i=1}^{n+1}$, then both $\{x_i\}_{i=1}^{n}$ and $\{x_i\}_{i=2}^{n+1}$ are basic with respect to F and hence with respect to E. So the Annihilator Lifting Lemma 3.1 applies. Hence there are c_1 and c_2 each $\geqslant 0$ whose sum equals one and such that $c_1\mu - c_2 v \in F^{\perp}$. Since $\mu \in H(x_1, ..., x_n)$ and $v \in H(x_2, ..., x_{n+1})$, we have that $c_1\mu - c_2 v \in H(x_1, ..., x_{n+1})$.

To prove (b) let $\{x_i\}_{i=1}^{n+1} \in \Delta_{n+1}$. Since E is an $n-1$ dimensional strong oscillating space there are $\mu \in Q(x_1, ..., x_n) \cap E^{\perp}$ and $v \in Q(x_2, ..., x_{n+1}) \cap E^{\perp}$. Since F is a Haar space every set of n points is basic for F, and therefore E. The annihilator lifting theorem applies, and there are c_1 and c_2 each $\geqslant 0$ whose sum equals one and such that $c_1\mu - c_2 v \in F^{\perp}$. Since F is a Haar space neither μ nor v by itself, annihilates F. Hence each of c_1 and c_2 is positive. We have that $c_1\mu - c_2 v \in Q(x_1, ..., x_{n+1})$. ∎

COROLLARY 3.3. $F_1 \subset F_2 \subset \cdots \subset F_n$. *If F_1 is the space of constant functions, and each F_i is a Haar space of dimension i, then each F_i is a strong oscillation space.*

4. MARKOV WEAK HAAR AND MARKOV OSCILLATING SPACES

THEOREM 4.1. *F is an oscillating space if and only if it is a weak Haar space that contains subspaces $F_1 \subset F_2 \subset \cdots \subset F$ such that F_1 is the constant, and each F_i is a weak Haar space of dimension i.*

Proof. If such subspaces exist it follows from the Lifting Theorem 3.2 that each F_i and F is an oscillation space. To prove the other direction we will show that if F is an oscillation space, then F contains an $n-1$ dimensional oscillation space. First we need:

Notation. Let $\|\cdot\|$ represent any norm on the finite dimensional space F. The dual space norm will also be written $\|\cdot\|$. We can assume that there is no $x \in \mathbf{X}$ such that F is constant on $(-\infty, x) \cap \mathbf{X}$. Let $m = \inf\{x \in \mathbf{X}\}$ and $w = \inf\{x \in \mathbf{X} - \{m\}\}$. Now choose u_i and $v_i \in \mathbf{X}$ such that (1) $u_i < v_i$, (2) if $m \in \mathbf{X}$ then $u_i = m$ otherwise $u_i \to m$, (3) if $m < w \in \mathbf{X}$ then $v_i = w$ otherwise $v_i \to w$, and (4) there exists a $g_i \in F$ such that $g_i(u_i) \neq g_i(v_i)$.

Proof Continued. $\alpha(u_i, v_i)/\|\alpha(u_i, v_i)\|$ has a subsequence (which we can assume we already have) that converges to a norm one functional L. $N = N(L)$, the null space of L, has dimension $n-1$, and for each $f \in N$ there is an $f_i \in N_i$ the null space of $\alpha(u_i, v_i)$ such that $f_i \to f$ (in norm, and in particular, pointwise). So if no member of N_i has an oscillation of length n on $[v_i, \infty) \cap \mathbf{X}$, then no $f \in L$ can have an oscillation of length n on $\mathbf{X}$. This property of N_i is precisely the Generic Subspace Lemma 2.4. ∎

THEOREM 4.2. *A weak Haar space is a weak Haar Markov space.*

Proof. We want to show that if F is a weak Haar space, then F contains an $n-1$ dimensional weak Haar space. This only requires a simplification of the above proof for Theorem 4.1. Define m and u_i as above and assume the existence of $g_i \in F$ for which $g_i(u_i) \neq 0$. Let $L = \lim(\hat{u}_i/\|\hat{u}_i\|)$ (where we have passed to a convergent subsequence if necessary). To show that the null space of L is a weak Haar space it suffices to show that the null spaces of $\hat{u}_i$ are weak Haar spaces on $(u_i, \infty) \cap \mathbf{X}$. Again this is the Generic Subspace Lemma 2.4. ∎

5. MARKOV CHEBYSHEV SPACES

THEOREM 5.1. *A Chebyshev space F on $\mathbf{X}$ contains an $n-1$ dimensional Chebyshev subspace E if and only if F can be extended to be a Chebyshev set on $\mathbf{X} \cup \{p\}$ for some point $p \notin \mathbf{X}$.*

Proof. If F can be extended then the null space of $\hat{p}$ is an $n-1$ dimensional Chebyshev subspace of F. Conversely if such a subspace E exists, there is a nonzero $L \in F^* \cap E^{\perp}$ (if $n > 1$), and we define $f(p) = L(f)$ for all $f \in F$ and some fixed $p \notin \mathbf{X}$. Then F is Chebyshev on $\{p\} \cup \mathbf{X}$. ∎

COROLLARY 5.2. *A Chebyshev space, F, on a countable set, $\mathbf{X}$, is a Markov Chebyshev space.*

Proof. For any set $n-1$ points $\{x_i\}_{i=1}^{n-1} \in \mathbf{X}$ the $n-1$ dimensional space, $\operatorname{span}\{\hat{x}_i\}_{i=1}^{n-1}$ is nowhere dense in the n dimensional space F^*. Since $\mathbf{X}$ is countable there are only countably many such spans. So from the Baire Category theorem there is an $L \in F^*$ that is not in the span of any $n-1$ point evaluation functionals. Let p be any point not in $\mathbf{X}$ and extend F by defining $f(p) = L(f)$. Now for any set of $n-1$ points $\{x_i\}_{i=1}^{n-1} \in \mathbf{X}$, $\{\hat{p}, \hat{x}_1, ..., \hat{x}_{n-1}\}$ are linearly independent, and so F is Chebyshev on $\{p\} \cup \mathbf{X}$. ∎

6. Markov Haar and Markov Strong Oscillating Spaces

Definition. $Y \subset \mathbf{X}$ is an *end set* (of $\mathbf{X}$) if $(\inf\{x \in \mathbf{X}\colon x \notin Y\}, \sup\{x \in \mathbf{X}\colon x \notin Y\}) \cap Y = \varnothing$. Further $\{a, b\} \subset \mathbf{X}$, $a < b$ is a *two point end set* if either

(i) $a = \min \mathbf{X} < b = \min\{\mathbf{X} - \{a\}\}$, or

(ii) $a = \max\{\mathbf{X} - \{b\}\} < b = \max \mathbf{X}$.

A measure μ supported on an end set is called an *end measure.*

We are interested in characterizing strong oscillating spaces, in the fashion Theorem 4.1 characterizes oscillating spaces as Markov weak Haar spaces. If F is oscillating and $\mathbf{X}$ contains a two point end set $\{a, b\}$ of the two smallest members of $\mathbf{X}$ then the construction in the proof of Theorem 4.1 produces the null space of $\alpha(a, b)$ as an $n-1$ dimensional oscillating subspace of F. The problem in the present setting is that even if F is Haar, the null space of a two point end measure is never a Haar space. Furthermore, as the next example shows, these are not the only oscillating subspaces of strong oscillation spaces that are not Haar subspaces.

Example 6.1. Let $\mathbf{X} = \{0, 1\} \cup [2, \infty)$, let F be the restriction to $\mathbf{X}$ of the polynomials of degree 3 or less, and let $E = \operatorname{span}\{1, x, x(x-1)(x-2)\}$. To show that E is 3 oscillating we use the fact that $x(x-1)(x-2)$ is convex for $x > 2$ implies that a linear function can maximize it on at most a single finite interval in $[2, \infty)$.

In Example 6.1, E is the null space of the end measure $\hat{0} - 2\hat{1} + \hat{2}$. The next theorem shows that these are the only types of examples.

Theorem 6.2. *Let F be a Haar space. If E is an $n-1$ dimensional weak Haar* (*oscillating, respectively*) *subspace of F that is not Chebyshev, then E is the null space of an end measure $\mu \in S(x_1, ..., x_k)$* ($H(x_1, ..., x_k)$, *resp.*).

Proof. Suppose that $f \in E$ has $n-1$ zeros $\{x_i\}_{i=1}^{n-1}$. We will shw there is a measure supported on these zeros that satisfies the theorem. If y is any point in $\mathbf{X}-\{x_i\}_{i=1}^{n-1}$ then since E is $n-1$ weak Haar (oscillating, resp.) there is a measure $\mu \in S(\{y\} \cup \{x_i\}_{i=1}^{n-1})$ $(H(\{y\} \cup \{x_i\}_{i=1}^{n-1})$, resp.) that annihilates E. Since F is a Haar space $f(y) \neq 0$, and so by Lemmas 1.1 and 1.2, the coefficient of $\hat{y}$ must be zero. It remains to show that $\{x_i\}_{i=1}^{n-1}$ is an end set. Suppose there is also a $z \in \mathbf{X}-\{x_i\}_{i=1}^{n-1}$ and $y < x_k < z$. Then there is also a measure ν in $S(\{z\} \cup \{x_i\}_{i=1}^{n-1})$ $(H(\{z\} \cup \{x_i\}_{i=1}^{n-1})$, resp.) that annihilates E. Again the coefficient of $\hat{z}$ is zero. So both μ and ν are supported on $\{x_i\}_{i=1}^{n-1}$. Notice also that each point in $(y, z) \cap \{x_i\}_{i=1}^{n-1}$ (in particular x_k) has μ and ν coefficients with weakly opposite sign. By the Annihilator Lifting Lemma 3.1 if $g \in F-E$, then (since F is Haar) sign $\mu(g) =$ sign $\nu(g)$. Therefore $\mu - c\nu \in F^{\perp}$ for some $c > 0$. Since F is Haar, no annihilator can be supported on n or fewer points. Hence $\mu - c\nu = 0$. But since the μ and ν coefficients of x_k have weakly opposite sign they must both be zero. This shows that μ is supported on a subset of $\{x_k\}$ that is an end set. ∎

Two observations come from the proof above.

COROLLARY 6.3. *Let F, E, and μ be as in the statement of the theorem.*

(a) *μ is supported on an end set of $n-1$ or fewer points.*

(b) *If $f \in E$ has $n-1$ zeros, then f vanishes on an end set.*

EXAMPLE 6.4. Let $\mathbf{X}$ and F be as in Example 6.1 above. Let $E = \text{span}\{1, (x-2)^2, x(x-1)(x-2)\}$. Although E is the null space of $(\hat{0}-\hat{1}) - 3(\hat{1}-\hat{2})$, it is not weakly Haar. To see this let $g(x) = x(x-1)(x-2) - 6(x-2)^2$. Then g is negative for $x < 2$ and $3 < x < 4$. Also g is positive for $2 < x < 3$ and $4 < x$.

THEOREM 6.5. *Let F be a Haar space on a set* $\mathbf{X}$ *that contains neither its maximum nor its minimum, then*:

(i) *every $n-1$ dimensional weak Haar subspace of F is a Haar space, and*

(ii) *F is a Markov Haar space.*

Proof. Part (i) follows from Theorem 6.2 since $\mathbf{X}$ has no finite end sets. From Theorem 4.2, F is a Markov weak Haar space, and from part (i) each of the associated subspaces is a Haar space. ∎

THEOREM 6.6. *Let F be an oscillating Haar space on a set* $\mathbf{X}$ *that contains no two point end sets. Then*

(i) *F is a strong oscillation space,*

(ii) *every* $n-1$ *dimensional oscillating subspace of* F *is a Haar space, and*

(iii) F *is a Markov strong oscillating space.*

Proof. For part (ii) we use the fact that a nonzero measure in $H(x_1, ..., x_n)$ must have two points in its support. Hence an end measure in $H(x_1, ..., x_n)$ must contain a two point end set in its support. In view of Theorem 6.2, this proves (ii). From Theorem 4.1, F is a Markov oscillating space, and from part (ii) each of the associated subspaces is a Haar space. Furthermore from the Lifting Theorem 3.2 each of the subspaces and F itself is a strong oscillation space. ∎

EXAMPLE 6.7. There are oscillation-Haar spaces that are not strong oscillation spaces. To construct one we just take a finite set of points $x_1 < x_2 < \cdots < x_k$. Then choose a $\mu = \sum_{i=2}^{k} (-1)^i c_i \alpha_i$ with the following properties: all $c_i \geqslant 0$, $c_2 > 0$, $c_k > 0$, not all $c_i > 0$, but if $c_i = 0$, then $c_{i+1} > 0$. Then the null space of μ is an oscillation space (since $\mu \in H(x_1, ..., x_k)$), and a Haar space (since each $\hat{x}_i$ has a nonzero coefficient), but it is not a strong oscillation space (since $\mu \notin Q(\{x_i\}_{i=1}^{k})$). To give a specific example, let $\mathbf{X} = \{1, 2, 3, 4\}$. Let F be the space spanned by the 3 functions whose values on $\{1, 2, 3, 4\}$ are f_1: (1, 1, 1, 1); f_2: (1, 1, 0, 0); and f_3: (1, 0, 0, 1). This is the null space of $(\hat{2}-\hat{1}) - 0(\hat{3}-\hat{2}) + (\hat{4}-\hat{3})$.

7. NOTES

There is no accepted convention in the literature for naming the concepts we called oscillating, alternating, Haar, weak Haar, Markov, etc. Besides the names here being given other meanings there are other common names such as T-systems and WT-spaces (for weak Tchebyshev), alternating spaces, complete systems, and extended T systems.

Corollary 2.2, that a weak oscillation space contains the constants, has been proven under additional hypotheses. For example, suppose that F is an oscillating Haar space of continuous functions on $[a, b]$. If $1 \notin F$ then some member $u \in F$ is a best approximation to 1. Then $u-1$ has an alternation of length $n+1$, and so u has an oscillation of that length. This argument has been adapted to somewhat less restrictive conditions. Zwick [14], for example, showed that an oscillation space of continuous functions on $[a, b]$ contained the constants. However, the general theorem was not known, and in fact it has been a common hypothesis to explicitly assume that an oscillation space contained the constants.

The determinant characterization Theorem 2.5 of weak Haar spaces was first proved by Jones and Karlovitz [4] for a special case. The general form is due to Bastien and Dubuc [1]. The proofs approximate the weak Haar space by a Haar space, and prove the result there. A more direct proof was given in Zielke [12].

The version of the Annihilator Lifting Lemma 3.1 stated in the paper is exactly what we needed the three times it was invoked. However, the following form seems, to me, to have more intrinsic connection with this theory.

LEMMA (Annihilator Lifting Lemma). *Let* $F=\{f\}+E$, *and suppose that both* F *and* E *are weak Haar spaces. Let* $\{x_i\}_{i=1}^n$ *and* $\{y_i\}_{i=1}^n \in \Delta_{n+1}$. *Suppose that* $\dim E = n-1$ *on both* $\{x_i\}_{i=1}^n$ *and* $\{y_i\}_{i=1}^n$. *If* $\mu=\sum_{i=1}^n \mu_i \hat{x}_i \in S(x_1, ..., x_n) \cap E^{\perp}$ *and* $\nu=\sum_{i=1}^n \nu_i \hat{x}_i \in S(y_1, ..., y_n) \cap E^{\perp}$, *then* $\operatorname{sign} \mu(f) \cong \operatorname{sign} \nu(f)$.

Proof. This is the same proof as above with the added observation that since E is a weak Haar space the two resulting $(n-1)\times(n-1)$ determinants are nonzero and have the same sign. That is, we write the determinant for the basis elements $f_1, ..., f_{n-1}, f$ and the points $\{x_i\}_{i=1}^n$. One of the points, say x_k, is not independent of the rest with respect to E. Using elementary column operations we replace the kth column with $\mu f_1, ..., \mu f_{n-1}, \mu f$. This gives a new determinant that is equal to $(-1)^{k+1}\mu_k$ times the original. Also the kth column consists of all 0 entries except the last which is μf. To compute the determinant we expand about the kth column to obtain that the original determinant is equal to

$$(-1)^{k+1}\frac{1}{\mu_k}(-1)^{n+k}\mu(f)\det\{f_i(x_j)\}_{i=1,j=1,j\neq k}^{n-1,n}.$$

We do the same process for the points $\{y_i\}_{i=1}^n$ and the annihilator ν, and again reduce the $n\times n$ determinant to $(-1)^{n+1}\nu(f)(1/\nu_h)\det\{f_i(x_j)\}_{i=1,j=1,j\neq h}^{n-1,n}$ for an appropriate h. From the weak Haar property, the two original $n\times n$ determinants weakly have the same sign, and the two final $(n-1)\times(n-1)$ determinants have the same sign. By our choices of k and h both $(n-1)\times(n-1)$ determinants are nonzero, and both μ_k and ν_h are positive. We conclude that $\mu(f)$ and $\nu(f)$ weakly have the same sign. ∎

One direction of Theorem 4.1 states that a weak Haar normalized Markov space is an oscillation space. The result was proved with additional assumptions by Zielke [12] and Zwick [14]. Zielke proved this version in 1985 [11] using a Gauss kernel approximation from his 1979 Haar space version. In 1989, he and Schwenker found a proof that avoided the Gauss approximation [13].

The other direction of 4.1 is in Zielke [12] under the additional assumption (unnecessary) that the space contains the constants.

Versions of Theorem 4.2 that weak Haar spaces are weak Haar Markov are due to Stockenberg [8] and to Sommer and Strauss [7]. Another approach to this result is to mimic the observation that if $G \subset C^1[a, b]$ is an n-dimensional space then G is an oscillating space if and only if the $n-1$ dimensional (since $1 \in G$) space of the derivatives of G is a weak Haar space. Hence the space of derivatives is Markov weak Haar if and only if G is Markov oscillating. So if there is a finite measure μ on an arbitrary $\mathbf{X}$ such that μ was positive on the open subsets in the topology generated by F (for example, $F \subset C[a, b]$ and μ Lebesgue measure, or $\mathbf{X}$ is countable and μ gives positive measure to each atom). Then the space of integrals of F along with the constants is oscillating if and only if F is weak Haar. Having proved that oscillating spaces are Markov, that would give that F is weak Haar Markov.

In fact, considerable attention is paid to spaces generated by taking integrals. The body of results laid out in Karlin and Studden [5] for such differentiable weak Haar systems is one of the original reasons for interest in the spaces studied here.

Theorem 5.1 characterizing when Chebyshev spaces have Chebyshev subspaces of codimension 1 is due to Zielke [10]. The proof here, although a little simpler, was mainly included for two other reasons. First, its an annihilator proof and seemed to fit the spirit of the other results here. Second, Corollary 5.2 was previously only stated (see Zielke [12]) for finite sets $\mathbf{X}$.

Versions of Theorem 6.5 for Markov Haar spaces are attributed to Krein (unpublished), Nemeth [6], and Zielke [10]. The general form is due to Zalik [9] and another proof is in Zielke [12].

Theorem 6.6 for Markov strong oscillation spaces is proved in Zielke [12] under the additional assumptions that $1 \in F$ and that $\mathbf{X}$ has the property that if x and $y \in \mathbf{X}$ there is a $z \in \mathbf{X}$ such that $x < z < y$.

The methods for 6.6 provide conditions on $\mathbf{X}$ under which every oscillating Haar space is Markov. It does not identify all the Markov spaces. For example, if we start with a oscillation Haar space, the method picks out an oscillating subspace of codimension one. Then if the conditions tell us that every such subspace is Haar, the proof is completed. It may be true that even though the oscillating subspace picked out by the method is not Haar, there are some other oscillating subspaces that are Haar. Specifically let F be the polynomials of degree 2 or less restricted to $\mathbf{X} = \{0\} \cup [2, 3]$. The method of the paper selects the subspace spanned by 1 and $(x-1)^2$ which is oscillating but not Haar. Perhaps a better method would select the oscillating Haar space spanned by 1 and x.

Example 6.7 shows that there are oscillating Haar spaces that do not contain oscillating subspaces of codimension 1. This example is in Zielke [12, Chap. 8, Exercise 2].

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